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Solving a Class of Multiplicative Programming Problems via C-Programming

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Abstract. In this note we show that many classes of global optimization problems can be treated most satisfactorily by classical optimization theory and conventional algorithms. We focus on the class of problems involving the minimization of the product of several convex functions on a convex set which was studied recently by Kuno *et al.* [3]. It is shown that these problems are typical composite concave programming problems and thus can be handled elegantly by c-programming [4]–[8] and its techniques.

Key words: Multiplicative objective functions, c-programming, parametric programming, global optimization.

1. Introduction

Under consideration in this discussion is the class of optimization problems of the following multiplicative form:

$$\text{PROBLEM P:} \quad z^* := \min_{x \in X} g(x) := \prod_{j=1}^{p} f_j(x) \tag{1}$$

where p is a positive integer, X is a convex subset of $\mathcal{R}^n$, and $\{f_j\}$ are real valued, strictly positive, convex functions on X. Since in general the function g is not convex with x, such problems are typically global optimization problems.

In a recent paper, Kuno *et al.* [3] showed that optimal solutions to Problem P can be obtained by solving the following parametric problem:

$$\text{PROBLEM MP:} \quad z^\circ := \min_{x,\xi} g(x;\xi) := \sum_{j=1}^{p} \xi_j f_j(x) \tag{2}$$

subject to

$$x \in X, \xi \in \mathcal{R}^p \tag{3}$$

$$\prod_{j=1}^{p} \xi_j \geq 1, \quad \xi \geq 0. \tag{4}$$

Namely, it was shown that there exists an optimal solution to Problem MP, say (x^*, ξ^*), such that x^* is an optimal solution to Problem P. Kuno *et al.* [3] also discuss an algorithm based on outer approximation for solving the parametric problem, thus solving Problem P.

In the next section we briefly analyze this approach and then in Section 3 we present a more direct parametric approach.

2. Analysis

The relation between Problem P and Problem MP can be explained as follows. Define,

$$\text{PROBLEM MP}': \quad z' := \min_{x,\xi} g(x;\xi) := \sum_{j=1}^{p} \xi_j f_j(x) \tag{5}$$

subject to

$$x \in X \tag{6}$$

$$\xi_j = \prod_{\substack{i=1 \\ i \neq j}}^{p} f_i(x) = \frac{g(x)}{f_j(x)}, \quad j = 1, 1, \ldots, p. \tag{7}$$

Observe that by construction, any pair (x,ξ) satisfying (6)–(7) also satisfies $g(x;\xi) = pg(x)$, $\forall x \in X$, $\xi \in \mathcal{R}^p$ satisfying (7). Thus, if we let

$$\xi(x) := \left(\frac{g(x)}{f_1(x)}, \ldots, \frac{g(x)}{f_p(x)} \right), x \in X \tag{8}$$

we immediately obtain the following:

OBSERVATION 1.

Let x^* be any optimal solution of Problem P. Then, $(x^*,\xi(x^*))$ is an optimal solution for Problem MP′ and furthermore, $\xi_j(x^*) f_j(x^*) = z^*$, $\forall j = 1,\ldots,p$. Hence $z' = pz^*$. ◇

Let us now examine the relation between Problem MP and Problem MP′. We want to replace the p equality constraints given by (7) with a *single* functional constraint involving the parameter ξ, say $\varphi(\xi) = c$. What form should φ and c take so that the inclusion of $\varphi(\xi) = c$ in the model will yield the desired optimal solution and satisfy (7)?

Observe that if we multiply the p equations stipulated by (7), we obtain

$$\prod_{i=1}^{p} \xi_i = \prod_{i=1}^{p} \frac{g(x)}{f_i(x)} = g^{p-1}(x). \tag{9}$$

This suggests that we define

$$\varphi(\xi) := \prod_{j=1}^{p} \xi_j, \quad \xi \in \mathcal{R}^p \tag{10}$$

and

$$c := g^{p-1}(x) \tag{11}$$

in which case $\varphi(\xi) = c$ yields

$$\prod_{j=1}^{p} \xi_j = g^{p-1}(x) \tag{12}$$

which in turn is equivalent to

$$\prod_{j=1}^{p} \frac{\xi_j}{\sqrt[p]{g^{p-1}(x)}} = 1. \tag{13}$$

We therefore conclude that any solution (x,ξ) satisfying (7) will also satisfy (13).

Next, note that because the functions $\{f_j\}$ are strictly positive on X, it is clear that the functional constraint (4) is binding, therefore we can replace (4) by

$$\prod_{j=1}^{p} \xi_j = 1, \xi \geq 0. \tag{14}$$

So it follows from (13)–(14) that Problem MP and Problem MP′ are essentially equivalent, the only difference being that in the former the variable ξ is normalized so as to yield 1 in the right-hand side of its functional multiplicative constraint. This immediately yields the following:

OBSERVATION 2.

The pair (x*,ξ') is an optimal solution of Problem MP′ if and only if the pair (x*,ξ°) is an optimal solution of Problem MP, where

$$\xi_j^\circ = \frac{\xi_j'}{\sqrt[p]{(z^*)^{p-1}}}, \quad j = 1 \ldots, p. \tag{15}$$

Thus,

$$z^{\circ} = \frac{z'}{\sqrt[p]{(z^{*})^{p-1}}} = \frac{pz^{*}}{\sqrt[p]{(z^{*})^{p-1}}} = p\sqrt[p]{z^{*}}.\Diamond \tag{16}$$

In summary, although not explicitly stated by Kuno *et al.* [3], the approach based on the use of Problem MP as a framework for solving Problem P has its origin in the substitution of variables stipulated by (7). In this sense, from the methodological point of view, Problem MP′ should be regarded as a more appropriate framework than Problem MP.

In the next section we show that Problem P can be treated by classical optimization methods and algorithms.

3. Quick and Elegant via C-Programming

The composite structure of the objective function g of Problem P immediately suggests the use of c-programming for the derivation of a suitable parametric problem. Details concerning c-programming and its applications can be found in [4]–[8]. In particular, in [8] its potential role in global optimization is discussed. Here we shall outline the general nature of the parametric problem that it proposes for Problem P.

Let us begin by recalling that c-programming was designed primarily for problems that admit the representation:

$$\text{PROBLEM C:} \quad \pi^{*} := \min_{y \in Y} c(y) := \Phi(u(y)) \tag{17}$$

where c is a real valued function on some set Y; u is a function on Y with values in $\mathcal{R}^k$; Φ is a real valued function on $u(Y)$: = $\{u(y: y \in Y\}$ and Φ *is differentiable and pseudoconcave with respect to u on some open convex set* $U \subseteq \mathcal{R}^k$ *such that* $u(Y) \subseteq U$.

The parametric problem, obtained by *linearizing* Φ with respect to u, takes the form:

$$\text{PROBLEM C}(\lambda): \quad \pi(\lambda) := \min_{y \in Y} c(y;\lambda) := \lambda u(y), \quad \lambda \in \mathcal{R}^k \tag{18}$$

where λ is a row vector and $\lambda u(y)$ denotes the inner product of λ and $u(y)$.

As expected, the relationship between the optimal solutions of Problem C and those of Problem C(λ) are linked via the *gradient* of Φ with respect to u; namely:

FUNDAMENTAL THEOREM of C-PROGRAMMING [7].

Let y^* be any optimal solution of Problem C and let λ^* denote the gradient of Φ with respect to u at $u(y^*$. Then any optimal solution of Problem C(λ^*) is also optimal with respect to Problem C. In other words, $Y^*(\nabla\Phi(u(y^*))) \subseteq Y^*$, $\forall y^* \in$

Y^* where Y^* denotes the set of optimal solutions of Problem C; $Y^*(\lambda)$ denotes the set of optimal solutions of Problem C(λ), and $\nabla\phi(u(y)) :=$ grad $\phi(z)|_{z=u(y)}$, $y \in Y$. □

Clearly, the Fundamental Theorem of C-Programming is based on such classical concepts as *linearization* and properties of *differentiable pseudoconcave functions*.

It is left to show that Problem P is indeed subject to this basic theorem. We can do this directly, observing that the function Ψ defined by $\Psi(z) := \prod_{j=1}^{p} z_j$, $z \in \mathcal{R}^p$ is differentiable and pseudoconcave on the open convex set $(0, \infty)^p$.

So Problem P satisfies all the requirements of c-programming and thus can be solved via the parametric problem

$$\omega(\lambda) := \min_{x \in X} \sum_{j=1}^{p} \lambda_j f_j(x), \quad \lambda \in \Gamma \tag{19}$$

where Γ is any subset of $\mathcal{R}^k$ such that $\nabla\Phi(u(x)) \in \Gamma$ for all x in X.

Thus, for each value of λ, the parametric problem is a classical optimization problem involving the minimization of a convex function. With regard to Γ, it should be pointed out that because the gradient of Φ is strictly positive everywhere on u(X), the elements of Γ can be normalised in the usual way.

In any case, the important thing to observe is that unlike Problem PM, where the parameter ξ appears both in the objective function and in the constraints, and where the objective function is optimized with respect to both the original variable x and the parameter ξ, in (19) the parameter λ does not appear in the constraints, nor do we have to optimize the objective function with respect to it. In short, we can employ standard Lagrangian methods to solve (19) for all λ in Γ. We shall not elaborate on this point here, but rather refer the reader to [4]–[8] for details concerning c-programming algorithms.

To appreciate the implications of these observations, let us consider the case where X = $\{x \in \mathcal{R}^n : Ax \geq b, x \geq 0\}$ and $f_j(x) = d_j x$, $d_j \in \mathcal{R}^m$, $j = 1,\ldots,p$, x $\in$ X, namely, the case where the constraints and the functions $\{f_j\}$ are linear.

In this case the parametric problem of c-programming is a *standard parametric linear programming problem*. In fact it can be solved by conventional LP algorithms designed for *multiobjective linear programming* problems. There are even readily available packages for this purpose [9]. It should be pointed out, though, that here we are not interested in generating all the non-dominated solutions, only the non-dominated *basic* solutions needed to cover Γ. Thus, solving the parametric problem of c-programming should be much faster than solving the usual multiobjective linear programming problem.

Furthermore, it should be recalled that the Fundamental Theorem of C-programming applies to a large class of composite functions, not only the multiplicative one. The only requirement is that Φ is differentiable and pseudoconcave on U with respect to u.

4. Summary and Conclusions

C-programming offers a classical framework for the analysis and solution of multiplicative programming problems of the type studied by Kuno *et al.* [3]. Furthermore, the same framework can be used as elegantly and as efficiently for the treatment of other classes of nonlinear optimization problems, namely problems whose objective functions admit the representation required by Problem C. It must be appreciated that because the solution set Y of Problem C is not required to satisfy any particular conditions, the Fundamental Theorem of C-Programming is valid even in cases where Y is a finite set. This means that c-programming offers a "classical approach" to the analysis and solution of many interesting and challenging *combinatorial* optimization problems.

On the algorithmic side, because the parametric problem of c-programming is of the classical Lagrangian variety, it would appear that efficient algorithms are in fact readily available for a number of important classes of multiplicative programming problems, e.g. multiplicative linear programming problems as well as multiplicative quadratic programming problems.

We should stress that these comments should not be interpreted as suggesting that we question the suitability of non-classical methods for the treatment of global optimization problems of the type discussed here. We merely indicate that classical methods and algorithms are capable of dealing with problems of this type, and that it may well be advantageous to use them for this purpose.

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